Comparison of Elasticity and Shell-Theory Solutions

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Comparisons of the stresses and displacements obtained by the use of three-dimensional elasticity theory and those predicted by several shell theories are presented for the problem of a long, circular-cylindrical shell subjected to a uniform, external, circumferential, radial line load. The classical shell theories of Timoshenko and Flügge and a Timoshenko-type shear deformation shell theory are used in making the comparison. Results are discussed for a ratio of inner to outer shell radii equal to 0.9, a Poisson's ratio of 0.3, and a transverse shear coefficient $\kappa = \frac{5}{6}$.

Introduction

Circular cylindrical shells often are used as basic structural elements and usually are analyzed by use of classical shell theories. These shell theories have been developed by many authors (see, for example, Refs. 1–3). The various approximations differ from one another essentially in the establishment of the strain-displacement relations. However, all have, with rare exception, made the following assumptions:

1) The ratio of the least radius of curvature of the middle surface to the thickness is large, i.e., $(a_0/h) \gg 1$.
2) The strain displacement relations are those of the classical theory of elasticity.
3) The normal radial stress is considered to be small with respect to the other normal stresses and is therefore neglected in the stress-strain relations.
4) Normals to the median surface of the undeformed shell remain straight and normal to the median surface after deformation; as a consequence, the transverse shearing strains vanish.

Recently, several papers[4–7] have been written in which the effect of the transverse shear deformation and in some cases also the effect of the normal radial stresses have been approximated. Often various shell results are compared with one another and with experimental data, and conclusions are reached concerning the relative accuracies of the theories.[8–10] It is not the purpose of this paper to verify experimentally the results obtained by use of one or another of these shell theories, but rather to investigate critically the assumptions made in developing these simplified theories from their foundations in the mathematical theory of elasticity.

The results of such a comparison, for the problem of a long circular cylindrical shell subjected to a uniform circumferential radial line load,[11, 12] are presented here and discussed briefly. Three shell theory solutions are used in the comparison, namely, the classical theories of Flügge and Timoshenko, and a Timoshenko-type shear deformation theory.[12] The latter is typical of the numerous theories that attempt to include the effect of transverse shear deformation and is the shell counterpart of the Timoshenko beam theory.[13] Upon neglecting the shear deformation terms, this theory reduces to the well-known Timoshenko shell equations. It is of interest to note that for the case of axisymmetry the Donnell equations reduce to the Timoshenko equations.

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Results

The shell geometry and loading are presented in Fig. 1, where coordinates and displacements are $r^* = ra$, $z^* = za$, $w^* = wa$, and $u^* = ua$, in which a is the outer radius of the cylinder and the unstarred quantities are nondimensional coordinates and displacements.

Normal Radial Stress

As a direct consequence of neglecting the normal radial stress component in the shell theories considered, the line load must be assumed to act as a transverse shear load. Once such theories are used, one is presented with a fait accompli. The question one must then ask is "How accurate are the results of these thin shell theories, notwithstanding their inherent misrepresentation of the support mechanism of transverse loads?" Furthermore, although by its very nature shell theory will result in inaccurate detailed descriptions of stresses and displacements in the neighborhood of transverse loads, it is necessary to determine the extent of the influence of such inaccuracies, i.e., whether such effects die out rapidly.

Thus, at the plane of the applied line load, for reasons previously noted, the shell theories will yield large transverse shearing stresses (which are discontinuous at $z = 0$) and negligible normal radial stresses, whereas the elasticity solution results in large normal radial stresses and zero transverse shearing stresses. However, if these two types of solutions are at all comparable, then, appealing to St. Venant's Principle, one could expect that the difference in the boundary conditions, induced by thin shell theory approximations, should be insignificant at a reasonably small distance from the load.

It is seen in Fig. 2 that the exact elasticity solution yields normal radial stresses σ_{rr} that are indeed negligible except near the plane where the concentrated load acts. At a distance $z = 0.2$ from the applied load, for example, the normal radial stresses already are reduced greatly. It is significant to note the excellent agreement of the solutions for $z \geqslant 0.2$. The Timoshenko normal radial stresses were calculated by satisfying the three-dimensional equilibrium equations[12] after obtaining the other stresses. The normal radial stresses were not calculated for the Flügge and shear deformation theory because of the impossibility of satisfying the conditions of zero stresses at the inner and outer shell radii simultaneously.

Transverse Shearing Stress

Except at $z = 0$, both the Timoshenko-type shear deformation and Timoshenko shell theories predict transverse shearing stresses σ_{rz} that are significantly less than those calculated from the elasticity solution (see Fig. 3). Moreover, at $z = 0.4$ and 1.0 the shell solutions predict shearing stresses having opposite signs to those computed from the elasticity solution.

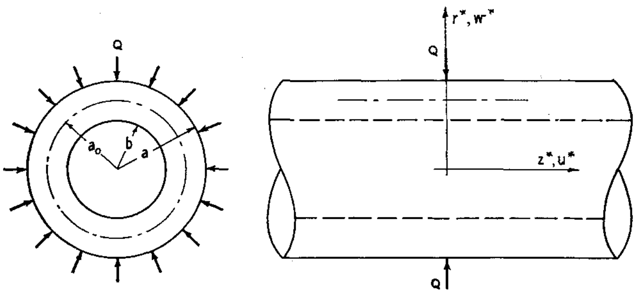

Fig. 1 Long circular cylindrical shell

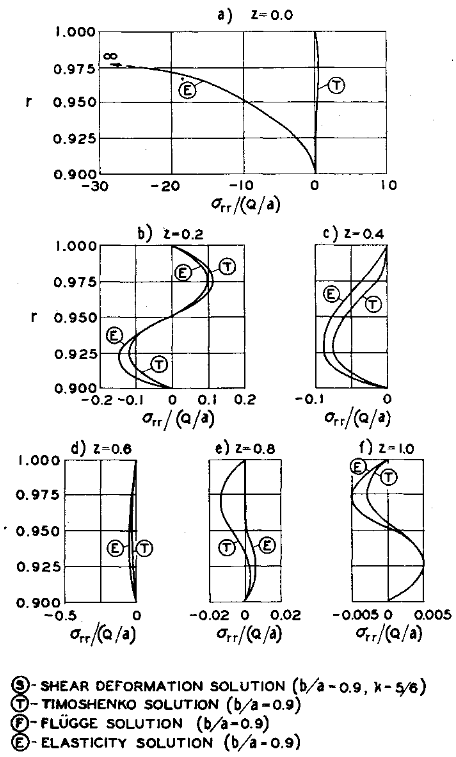

Fig. 2. Nondimensional normal radial stress, $\sigma_{rr}/(Q/a)$

The latter discrepancy can be explained by observing that the shearing stresses vanish in the vicinity of these planes. Thus a small variation in the deflections predicted by the shell theories easily can lead to such results.

The transverse shearing stresses were not calculated for the Flügge theory because of the inability to satisfy the zero stress requirements at both the inner and outer surfaces. An indication of the resulting stress field in the immediate neighborhood of the concentration can be obtained from the two-dimensional analog of the Boussinesq problem.[15] The solution reveals that the shearing stresses in the immediate vicinity of the load singularity are very large and are highly peaked close to the outer shell surface. As the plane $z = 0$ is approached, the peaking increases and moves closer to the outer surface. In the limit, one obtains the singularity at the outer surface at $z = 0$. From a physical point of view, it is fairly obvious that large shearing stresses must exist in the neighborhood of the load. These stresses are induced as a consequence of the necessity of establishing compatible radial displacements. At a sufficiently large distance from the load concentration ($z = 0.2$), the elasticity theory predicts an almost parabolic shearing stress distribution. Preliminary results obtained from an investigation of a problem quite similar to that presently being discussed very clearly reveal characteristics for the shearing stresses as described previously.

Normal Circumferential and Longitudinal Stresses

The normal circumferential stresses $\sigma_{\theta\theta}$ and the normal longitudinal stresses σ_{zz} predicted by shell theory compare quite favorably with those computed from the elasticity solution except in the neighborhood of the load singularity (see Figs. 4 and 5). Furthermore, they are indeed much greater than the radial normal stresses except in the neighborhood of the transverse load.

Radial Displacement

A comparison of the radial displacements (Fig. 6) reveals that the shear deformation shell theory yields excellent results. It predicts a deflection at the plane of the load ($z = 0$) which is 2% smaller than that predicted for the deflection at the median surface by the exact solution. The transverse shear

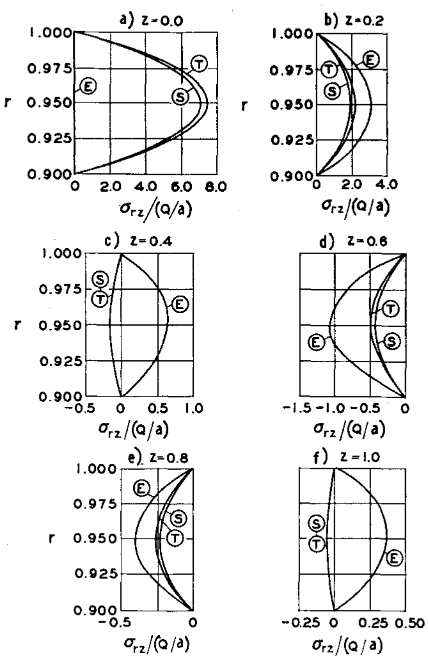

Fig. 3 Nondimensional transverse shearing stress, $\sigma_{rz}/(Q/a)$

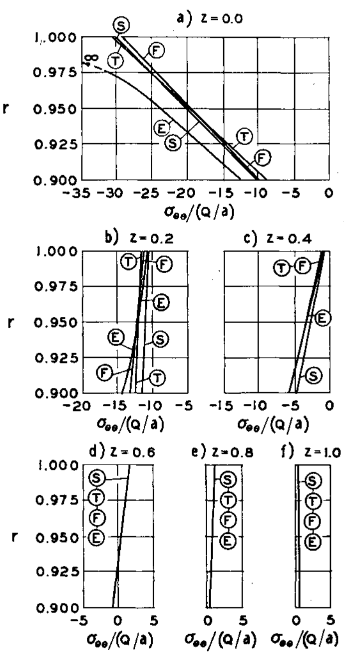

Fig. 4 Nondimensional normal circumferential stress, $\sigma_{\theta\theta}/(Q/a)$

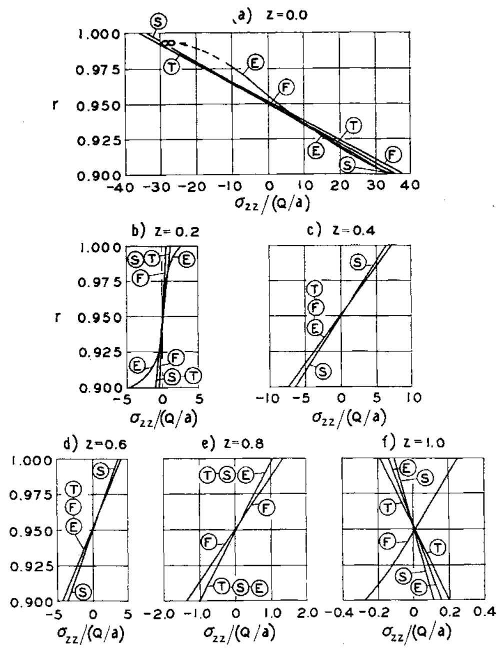

Fig. 5 Nondimensional normal longitudinal stress, $\sigma_{zz}/(Q/a)$

coefficient $\kappa = \frac{5}{6} = 0.833$, derived by Naghdi,[6] was used in the calculations of the shear deformation theory. It is interesting to note, moreover, that Herrmann and Mirsky[7] suggest the use of $\kappa = 0.86$, which for short wavelengths should be reduced to Mindlin's[14] value of $\kappa = (\pi^2/12) = 0.822$. This is additional justification for the use of Naghdi's coefficient.

The radial displacements predicted by the classical shell theories (Timoshenko and Flügge) are seen to be in good agreement with the values obtained from the elasticity solution. However, in the neighborhood of the plane of the load singularity, the classical shell theory solutions are about 8% too small. This is to be expected, since shearing strains have been neglected in the classical shell theory. In the immediate neighborhood of a concentrated load, the shearing stresses and strains are large (see discussion under "Transverse Shearing Stress") and increase the radial deflections appreciably. It is therefore not surprising to find that the inclusion of the shear deformation terms reduces the foregoing 8% underestimation to 2%.

The variation of the radial displacement through the shell thickness is less than 3% (Table 1, Ref. 11) even in the immediate neighborhood of the concentrated load. Of course, the results for the thin shell theories presented here have been obtained by assuming that the radial displacement is independent of the thickness coordinate.

Longitudinal Displacement

The longitudinal displacements u predicted by shell theory are slightly less than those calculated from the elasticity solution (see Fig. 7). Moreover, the Kirchoff-Love assumption of shell theory that normals to the median surface of the undeformed shell remain straight and normal to the median surface after deformation is seen to be good, as is evident when comparing the slopes of the radial displacement curve (Fig. 6) to the slopes of the longitudinal displacement curves (Fig. 7).

Conclusions

In conclusion, it can be stated that classical shell theories represent good approximations to the three-dimensional stress problem for relatively thick shells ($b/a = 0.9$, i.e., $h/a_0 = 1/9.5$). However, these shell theories do not predict adequately the stresses and radial deflections in the neighborhood of transverse localized loads. At these locations the radial deflections are predicted quite accurately by the transverse shear deformation shell theories. Interestingly enough, shear deformation theory does not improve the accuracy of the axial displacements and the stresses. It is felt that the inclusion of transverse normal strain terms[5] would improve the results significantly. Such an investigation is currently being carried on using loads distributed over finite widths and will be reported upon a later date.

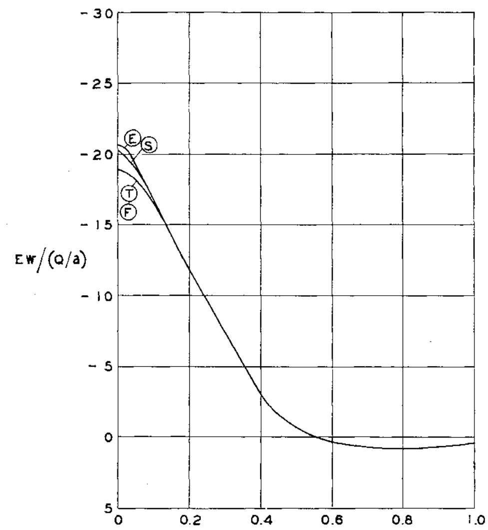

Fig. 6 Nondimensional radial displacement, $Ew/(Q/a)$

For the problem presented in this note, the classical shell theories predict deflections under the load which are 8% smaller than those obtained from the elasticity solution. Thus, if one would consider a ring-stiffened shell subjected to a uniform pressure, then the maximum error in the calculated interaction load may be as much as 8% and will occur when the spacing of the re-inforcing rings is large and the rings are rigid. As the ring spacing decreases and the ring rigidity decreases, the error is reduced. Thus, for ordinary design the interaction loads are not significantly different whether one uses classical shell theories or elasticity or shear

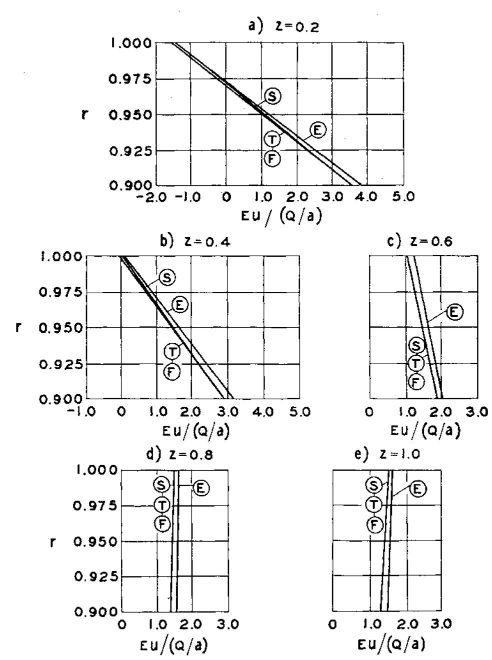

Fig. 7 Nondimensional longitudinal displacement, $Eu/(Q/a)$

deformation shell theories. What is significant, however, is the difference in the stress distribution, and it is rather with this consideration in mind that future work in elasticity theory pertaining to reinforced shell design should be directed.

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Postbuckling Behavior of Axially Compressed Circular Cylinders

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The postbuckling behavior of a circular cylindrical shell subjected to axial compression is considered. Because of numerical difficulties, previous analyses of this problem have been severely restricted and do not yield quantitatively satisfactory results. The most accurate such analysis gives a minimum postbuckling load that is approximately three times higher than corresponding test values. In the present analysis, the number of free constants in the displacement function has been increased successively until no significant change occurs in the magnitude of the minimum postbuckling equilibrium load. The load so computed is found to be in close agreement with available test results. Although the minimum postbuckling load may be considered as a lower bound for the buckling load, it is, in general, too conservative to be practically useful as a design limit. Hence, the present analysis is not directly useful for design. However, it shows that agreement between theory and tests can be obtained for the postbuckling behavior of axially compressed cylinders, and it indicates which terms are needed in the assumed form of the deflection pattern in the finite displacement analysis. The analysis also can be extended to include, for instance, the effect of initial geometrical imperfections.

Nomenclature

a	=	cylinder radius
a_{ij}	=	constants [see Eq. (7)]
E	=	Young's modulus
F	=	stress function
j,k	=	integers [see Eq. (7)]
L	=	cylinder length
l_x,l_y	=	half wavelengths in axial and circumferential directions, respectively
n	=	number of waves in circumferential direction
t	=	shell thickness
u,v,w	=	nondimensional displacement components of a point in the middle surface of the shell in the axial, circumferential, and normal directions, respectively; corresponding distances are au, av, and aw, and w is positive inward
V	=	total potential energy of shell
x	=	nondimensional coordinate in axial direction; corresponding distance is ax
δ	=	end shortening per unit length
δ_{CL}	=	σ_{CL}/E
$\epsilon_x,\epsilon_\phi,\gamma_{x\phi}$	=	strains at a point in the middle surface of the shell
$\sigma_x,\sigma_\phi,\tau_{x\phi}$	=	stresses at a point in the middle surface of the shell, nondimensional
σ	=	axial compressive stress
σ_{CR}	=	critical compressive stress
σ_{CL}	=	$\{E/[3(1-\nu^2)]^{1/2}\}(t/a)$
ν	=	Poisson's ratio
ϕ	=	angular coordinate

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Introduction

It became evident early in the history of the problem that the classical theory fails to produce results in agreement with tests for the buckling of a cylindrical shell under axial compression. Hence, the theory of finite deflections has been explored as a possible way to obtain theoretical results